Low sample size and regression: A Monte Carlo approach

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Abstract:

This article performs simulations with different small samples considering the regression techniques of OLS, Jackknife, Bootstrap, Lasso and Robust Regression in order to stablish the best approach in terms of lower bias and statistical significance with a pre-specified data generating process -DGP-. The methodology consists of a DGP with 5 variables and 1 constant parameter which was regressed among the simulations with a set of random normally distributed variables considering samples sizes of 6, 10, 20 and 500. Using the expected values discriminated by each sample size, the accuracy of the estimators was calculated in terms of the relative bias for each technique. The results indicate that Jackknife approach is more suitable for lower sample sizes as it was stated by Speed (1994), Bootstrap approach reported to be sensitive to a lower sample size indicating that it might not be suitable for stablish significant relationships in the regressions. The Monte Carlo simulations also reflected that when a significant relationship is found in small samples, this relationship will also tend to remain significant when the sample size is increased.

Keywords: *Small sample size, Statistical significance, Regression, Simulations, Bias*

JEL Classification: C15 – C19 - C63

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Content

Introduction 3

Some studies related 3

Methodology 6

Results 7

Statistical significance 7

Bias behavior of the parameters 10

Ordinary Least Squares 10

Jackknife 16

Bootstrap 20

Lasso regression 24

Robust regression 28

Comparing the estimations 32

Conclusions 35

Bibliography 37

Introduction

One situation that might happen while we're trying to analyze data and make empirical inferences over a phenomenon is that we may have a low (or reduced) number of observations. This is usually associated with the lack of confidence in the estimations, especially when we're opting for the regression analysis in the multivariate framework.

One answer to avoid this problem is to perform descriptive statistics and proceed with the deduction patterns, however one could ask: are we really sure that our estimations are unreliable (they really lack of confidence?) when we're regressing a model in the presence of low observations? Naturally, the literature supports this idea from different perspectives, as an example Bujang, Sa'at, & Tg Abu Bakar Sidik (2017) studies state that in order to obtain coefficients closer to the population parameters we need around 300 observations.

But if our phenomenon has not been studied (or documented) properly in order to obtain a significant number of observations, should we discard immediately the multiple regression technique to analyze it? The aim of this paper is to provide evidence that regression can have consistent estimates of the coefficients when we're dealing with low number of observations.

The methodology consists mainly in the use of Monte Carlo simulations derived from a linear data generating process -DGP- to perform conclusions about the bias of the estimated coefficients in the regression framework. The estimation techniques involve ordinary least squares -OLS-, Jackknife, Bootstrap, Robust Regression and Lasso approaches.

Some studies related

The number observations can be classified in general terms as it can be found in the study of Mason & Perreault (1991) where a sample size of 30 or lesser is considerate small, samples around 150 observations can be considered as moderate and finally, samples bigger than 250 or 300 are tagged as large. One interesting problem that arises in small samples are relative to the statistical inferences, in fact "using a sample smaller than the ideal increases the chance of assuming as true a false premise" (Faber & Fonseca, 2014). This implies considering the two types of errors in statistical hypothesis testing, the type I and II errors. In simple words, the first type of error refers that our null hypothesis H_0 (relative to a specific proposition) is true but we reject it, while the second type of error refers when our H_0 is false but we don't reject it.

Small sample size and incorrect inferences in the parameters' significance test are studied by Colquhoun (2014) indicating that a p-value lesser than 5% might not be statistical significant since

the results are derived from "underpowered statistical inferences". From this, the risk of using a small size would be the possibly type I error in the regression framework.

More from this idea can be found in another study of Forstmeier, Wagenmakers, & Parker (2017) where the problem of false-positive findings can be derived from a decreased sample size and incorrect p-values. Also, the problem of statistical inferences is correlated with the replication procedure, in other words, the last two types of errors seems to be sensitive to the number of replications in a way that the results derived from one inference might not match the result of a similar exercise concerning a similar set of data. This is a fair point in the analysis, the number of replications might affect the statistical inference and the overall converge rate to the population parameters of the estimations, so it should be taken in account. This idea lead to a basic statement: as we increase the number of replications of an experiment, we're getting closer and closer to the expected behavior of the population parameters in the inference.

This authors also make a valid point regarding some underlying assumptions of the estimations, for example autocorrelation, correct specifications, no omitted variables in general. In this case, small sample size inferences can be harmful where also the ordinary least squares assumptions are not satisfied.

A remarkable study performed by Holmes Finch & Hernandez Finch (2017) start by analyzing tools like Lasso, Elastic net, Ridge regression and the Bayesian approach regarding the situation when we got high dimensional multivariate data relative to an even bigger number of variables. In this case, the number of independent variables maybe close or equal to the sample size, yielding in unstable coefficients and standard errors (this ones are needed to the formulation of the hypothesis testing procedure) (Bühlmann & Van De Geer, 2011). The result of this experiments tends to demonstrate that regularization methods, in particular the ridge regression approach where more accurate in terms to control bias and type I errors produced in the estimations with low sample data for multiple regression analysis.

Speed (1994) tries to contribute to the solution of low sample size in the regression framework, considering sample reuse validation techniques. These techniques refer to the Jackknife and Bootstrap approaches related to the multiple regression estimation. An important statement of this author is:

> "Researchers should note that the overwhelming case is that reduction in sample size is far more likely to reduce the likelihood of finding any significant relationships than to increase it. This is due to the way that sample size affects test power. The researcher sets the level of type I error (the probability of accepting a hypothesis when false in reality) in any test, normally at 0-05, and critical values calculated for the given size of sample. Small sample sizes are no more likely to result in wrongfully claiming a relationship exists than is the case for larger samples." (Speed, 1994, pág. 91)

This interpretation is indeed useful since it states that low sample relationships are more likely to be found when the sample size increases over the experiments. In fact, there are some literature which also critiques the role of large samples in the estimations, arguing that anything becomes significant. Within this idea we can find the study of Lin, Lucas Jr. & Shmueli (2013) where they affirm that as the sample size is increasing, the p-value starts to decrease drastically to 0, which could lead to statistical significant results which are not sensitive over the regression analysis. Meanwhile a low sample size is more sensitive to the correlation between the variables (this implies sensibilization to the changes too) leading to think that large sample size might find significant results when it's just an overwhelming product of the power of the sample without accurately indicating real (or strong) relationships among the variables. In fact, Faber & Fonseca (2014) appoints that samples cannot be either too big or too small in order to perform statistical inferences.

Up to this point we're facing problems on both sides of the sample size, too much can be misleading and unsensitive to true relations among the variables (which can be specially the case in the regression analysis) and on the opposite, when we got a little sample size, we might have results that are inconsistent across replications driving to errors of type 1.

Methodology

The main idea of the methodology is to perform Monte Carlo approximations across different types of estimations which involves OLS, Jackknife, Bootstrap, Lasso and Robust Regression, assuming a multivariate data generating process in a linear form as it follows:

$$y_i = \alpha + \gamma x_{1,i} + \delta x_{2,i} + \theta x_{3,i} + \vartheta x_{4,i} + \varphi x_{5,i} + u_i \quad (1)$$

Equation (1) is calibrated setting the population parameters α, γ, δ, θ, ϑ, φ as all equal to 10 for the i observations. The objective is to identify which of the estimation types suits better in terms of accuracy of the estimators. In this case, across the simulations it is assumed that

$$x_j \sim N(0,1) \quad , \quad u_i \sim N(0,1) \quad (2)$$

From (1) we're setting the number of replications to 10, 100 and 500 while the number of observations would be set as first to 6 in order to induce on purpose the micronumerosity phenomenon and see how the estimators react to this problem, also the other number of observations are set to 10, 20 and 500. There's no need to test a higher number of observations since empirical literature has established that overall significance and unbiasedness is influenced by a large sample size. The relative bias of the estimators among the coefficients would be expressed as a relative difference from the population parameter, following a general idea that:

$$Bias = \left| \frac{\beta_j - \overline{\beta_J}}{\beta_j} \right| \quad (3)$$

Where β_j represents the parameter of the j variable contained in equation (1) and $\overline{\beta_j}$ represents the estimated parameters. The overall bias can be expressed in terms of expected values as it follows:

$$O.B. = \left| \frac{\beta_j - E(\overline{\beta_J})}{\beta_j} \right| \quad (4)$$

Where the mean value of the estimated parameters would be our expected value $E(\overline{\beta_J})$ of the coefficients by each type of regression. In this case, the bias would be expressed in terms of percentage, indicating that 0 would be closer to a perfect match with the true parameter.

In order to see change in the statistical significance of the coefficients, single Monte Carlo simulations would be presented in the usual regression output for each type of estimation (OLS, jackknife, bootstrap, lasso and robust regression) with the different size in observations as mentioned before, then the bias results would be presented for each type of estimation discriminated by size of the sample and number of replications.

Results

Statistical significance

The OLS simulation practiced, establish that the pattern of statistical significance for all estimators will remain as long as the sample size is increasing, the special case of micronumerosity tend to disrupt the statistical significance as expected, but the yielding estimators seems to be closer to the DGP.

Table 1 OLS Monte Carlo simulation with different sizes

	(1)	(2)	(3)	(4)
VARIABLES	y	y	y	y
x1	9.549	9.288***	9.915***	9.991***
	(0)	(0.440)	(0.208)	(0.0468)
x2	10.36	9.915***	10.01***	9.961***
	(0)	(0.491)	(0.200)	(0.0499)
x3	8.952	9.709***	10.27***	9.979***
	(0)	(0.362)	(0.211)	(0.0457)
x4	10.66	10.44***	9.977***	10.04***
	(0)	(0.295)	(0.207)	(0.0453)
x5	10.70	9.233***	10.59***	10.02***
	(0)	(0.530)	(0.289)	(0.0506)
Constant	8.902	9.979***	10.10***	9.997***
	(0)	(0.394)	(0.225)	(0.0463)
Observations	6	10	20	500
R-squared	1.000	0.999	0.999	0.998

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Source: Own elaboration

As an interesting thing to consider, the R^2 values changes when we estimate the DGP with 20 observations, to a lower accuracy (but still closer to 1) in the context of 500 observations, this is proof that the property of consistency among the OLS estimator is achievable (and of course all classical assumptions of the linear regression model are also satisfied). This tends to indicate that the affirmation of Speed (1994) regarding to the relationships found in small sample size tend to remain as the size of the sample increases.

Going further with the jackknife estimation, it can be observed that it cannot be computed in the presence of perfect micronumerosity, leading to the impossibility to even approach to get a result from observed coefficients, among the statistical significance it also remains across sample size, suggesting the same result from OLS.

Table 2 Jackknife estimation with different sample size

VARIABLES	(1) y	(2) y	(3) y	(4) y
x1	-	9.733***	10.25***	10.00***
	-	(0.470)	(0.358)	(0.0445)
x2	-	9.892***	9.891***	9.926***
	-	(0.296)	(0.380)	(0.0454)
x3	-	10.42***	10.33***	10.02***
	-	(0.667)	(0.296)	(0.0445)
x4	-	11.04***	10.09***	9.977***
	-	(0.523)	(0.403)	(0.0476)
x5	-	10.29***	9.627***	10.03***
	-	(0.784)	(0.401)	(0.0434)
Constant	-	9.454***	9.830***	10.04***
	-	(0.433)	(0.282)	(0.0462)
Observations	6	10	20	500
R-squared	-	0.999	0.998	0.998

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Source: Own elaboration

The goodness of fit of the model tends to be reduced as the sample size is increased considering this type of estimation, we can also see that the coefficients vary from the ones estimated via OLS.

The bootstrap estimation is presented in the table and display results a little bit different from the OLS and the jackknife, in the induced model with micronumerosity the coefficients can be computed, however, standard errors cannot be estimated.

Table 3 Bootstrap estimation with different sizes

VARIABLES	(1) y	(2) y	(3) y	(4) y
x1	10.06	10.26**	9.651***	10.01***
	(0)	(4.026)	(0.214)	(0.0512)
x2	9.375	10.67**	10.23***	9.958***
	(0)	(4.408)	(0.264)	(0.0359)
x3	10.85	9.744***	10.40***	10.02***
	(0)	(2.750)	(0.200)	(0.0379)
x4	10.24	10.27	9.718***	10.05***
	(0)	(8.483)	(0.177)	(0.0422)
x5	10.68	10.15***	9.959***	10.01***
	(0)	(3.871)	(0.238)	(0.0479)
Constant	11.34	10.50***	10.04***	9.993***
	(0)	(2.564)	(0.286)	(0.0396)
Observations	6	10	20	500

R-squared	1.000	0.993	0.999	0.998

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Source: Own elaboration

According to the Monte Carlo experiment with the bootstrap technique, it can be seen that as the sample size is increasing, the statistical significance will also be increased. The variables x_1, x_2 and x_4 demonstrate this situation, where for example with n=10, for x_4 there wasn't a statistically significant relation with y in the regression model. Then as soon as we increased the sample size to n=20 the variable turned to be significant, the similar case can be observed with x_1 and x_2 where they only were significant at a 5% with n=10. Then with n=20 they become significant at 1%, indicating that the bootstrap approach is sensitive to number of observations regarding to the coefficient hypothesis testing. Which might suggest is not a good idea to perform this technique with low sample size since it might discard a real relationship among the variables.

Following with the Lasso regression, micronumerosity doesn't allow the estimation of the coefficients. And the overall statistical significance remains equal across regressions with different sample sizes. This result indicate that estimations are consistent across models using the right variables with the specific function formal equally to the DGP.

Table 4 Lasso estimations with different sizes

	(1)	(2)	(3)	(4)
VARIABLES	y	y	y	y
x1	-	11.18***	10.09***	10.03***
	-	(0.832)	(0.242)	(0.0453)
x2	-	9.605***	9.785***	9.913***
	-	(0.376)	(0.272)	(0.0452)
x3	-	10.70***	9.866***	10.07***
	-	(0.551)	(0.264)	(0.0442)
x4	-	9.441***	9.585***	9.873***
	-	(0.355)	(0.235)	(0.0421)
x5	-	10.17***	9.861***	9.984***
	-	(0.397)	(0.336)	(0.0433)
Observations	6	10	20	500

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Source: Own elaboration

It can be noted that Lasso regression omits the constant parameter, but the highest possible good of fit has been selected according to the variables. Thus, the statistical significance of the estimators prevails across the models with different sample sizes. However, it is necessary to appoint that Lasso regression doesn't look directly at the p-values or the standard errors, since its sole objective is to isolate a model where the predictions becomes more suitable according to the data (StataCorp, 2019).

The robust regression estimates are similar to the ones done with Lasso and jackknife in terms that the model cannot be estimated when micronumerosity is present. The other results related to the statistical significance of the estimators indicate that when we're in the context of short samples, the relationships remain significant as the number of observations increase.

Table 5 Robust Regression with different sizes

	(1)	(2)	(3)	(4)
VARIABLES	y	y	y	y
x1	-	9.866***	9.883***	10.02***
	-	(0.443)	(0.282)	(0.0442)
x2	-	11.04***	9.560***	9.978***
	-	(0.618)	(0.255)	(0.0460)
x3	-	10.15***	10.35***	10.02***
	-	(0.611)	(0.290)	(0.0415)
x4	-	9.315***	10.16***	9.972***
	-	(1.361)	(0.333)	(0.0441)
x5	-	10.88***	10.25***	9.963***
	-	(0.649)	(0.215)	(0.0430)
Constant	-	10.58***	9.949***	10.02***
	-	(0.935)	(0.226)	(0.0444)
	-			
Observations	6	9	20	500
R-squared	-	1.000	0.999	0.998
Standard errors in parentheses				
*** p<0.01, ** p<0.05, * p<0.1				

Source: Own elaboration

An interesting thing to appoint is that as long as we're having a large sample regarding our regressions, the goodness of fit tends to be reduced somewhat across estimations. This led to the conclusion that R^2 is sensitive to the number of observations among the sample, in a really little inverse relationship.

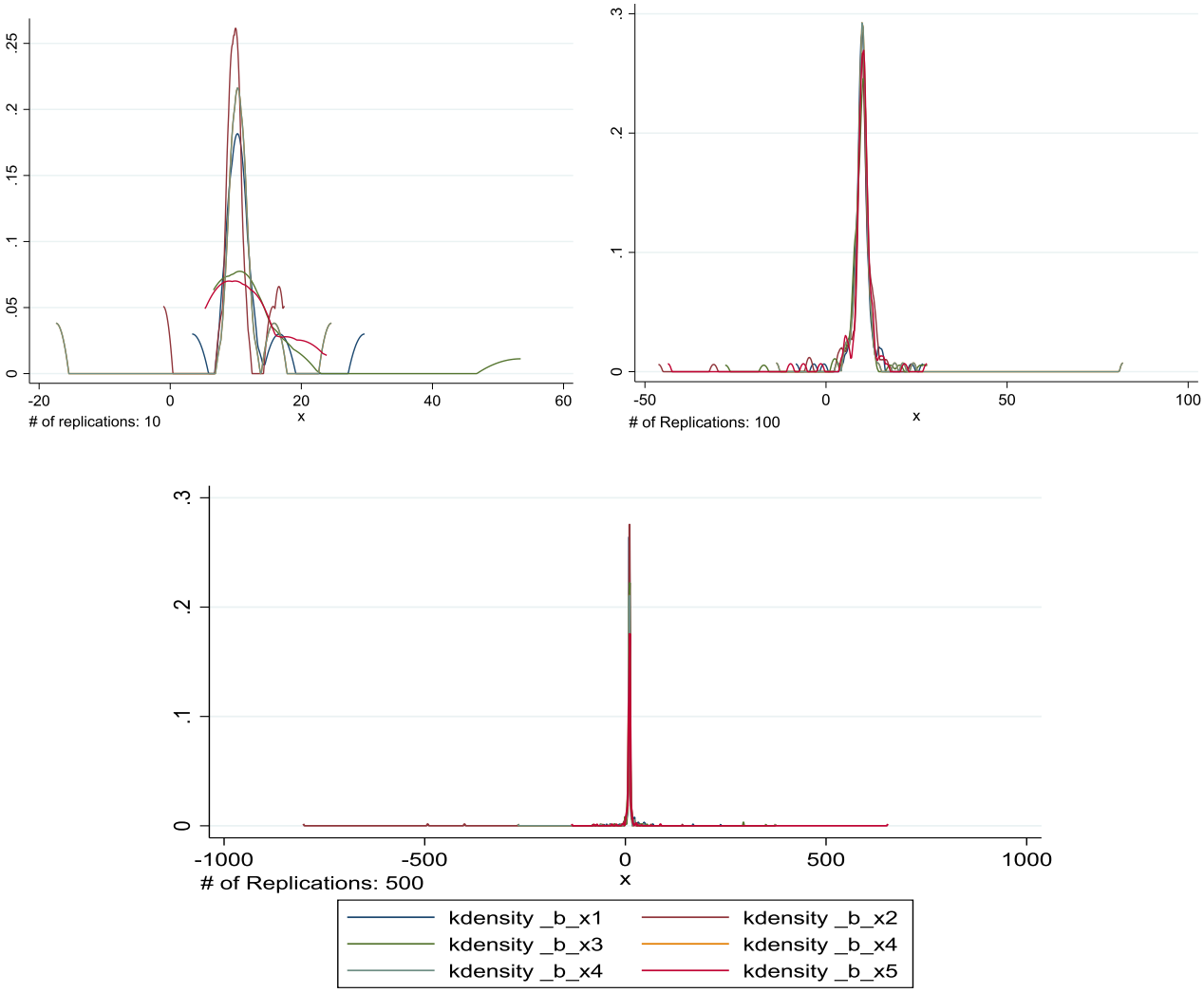
Bias behavior of the parameters

This section consists of the results for each type of estimation (OLS, jackknife, bootstrap, lasso and robust regression) referring to the distributions across replications for the coefficients, kernel densities were used for each coefficient of the different x variables in order to provide analysis regarding the importance of the number of replications.

Ordinary Least Squares

Considering a number of 6 observations, the coefficients for each variable tend to be somewhat unstable when the number of replications is low, meaning that in the presence of micronumerosity the estimators are less likely to be trustable. As replications are increased to 100 and 500, the estimators seems to converge to their true value of 10, the situation clearly implies that across regressions with random data, as long as we replicate enough times the experiments, the expected value seems to be close to our DGP, it should be noted that OLS estimators stills covers some extreme values which would be affecting the consistency across replications, as we can see it in the graphical pattern.

Graph 1 OLS - Distributions of the Coefficients with n=6



Source: Own elaboration

This results proofs evidence that under micronumerosity, OLS estimates are unstable so it should be avoided at all cost. Considering the 500 replications for the 6 observations regression with OLS, the descriptive statistics for each coefficient reflects an undeniable reality. The minimum and maximum values are out of scale regarding to our DPG where each coefficient equal to 10, even when the mean value is somewhat closer, the results yield unstable.

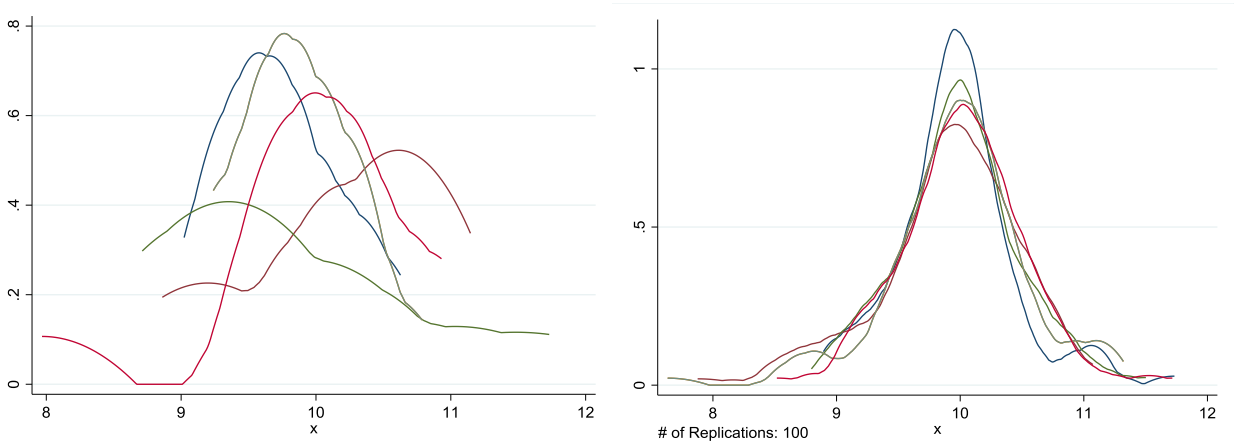
Table 6 OLS Descriptive Statistics with n=6

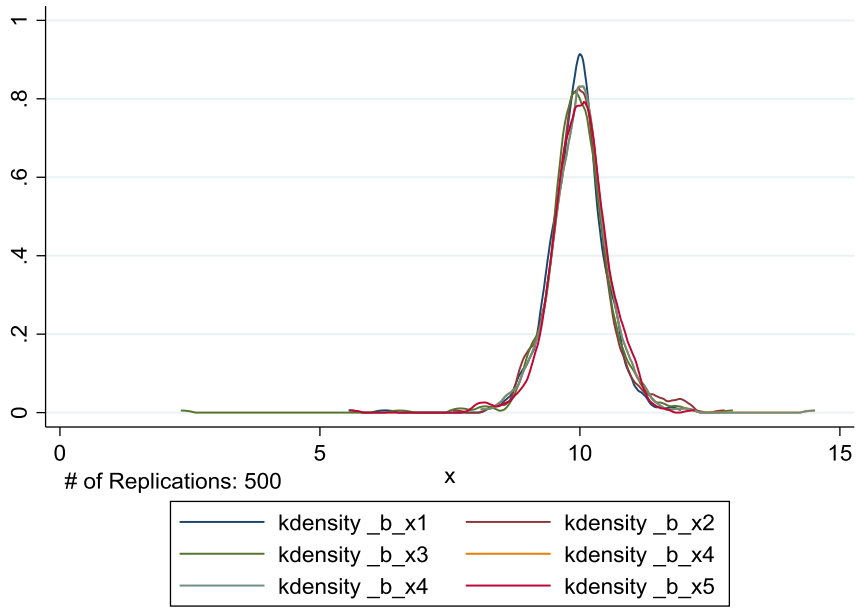
Variable	Replications	Mean	Std.Dev.	Min	Max
_b_x1	500	9.78	13.529	-86.594	243.693
_b_x2	500	13.807	102.362	-588.518	2199.746
_b_x3	500	7.444	49.54	-1044.307	199.705
_b_x4	500	10.553	28.372	-306.405	526.281
_b_x5	500	4.668	62.136	-1043.826	62.116
_b_cons	500	5.83	92.71	-2015.365	188.439

Source: Own elaboration

Now considering the number of observations as 10, the following pattern of distributions can be found:

Graph 2 OLS - Distributions of the Coefficients with n=10





Source: Own elaboration

There is a quick and stable rate of convergence relative to the distributions of the estimators for each variable which is depicted across replications. The distributions tend to be normal as the simulation number increase, leading to the true value of the estimators for all x variables and the constant term. The descriptive statistics are shown ahead considering 500 hundred replications of the Monte Carlo simulations with n=10 observations.

Table 7 OLS Descriptive Statistics n=10

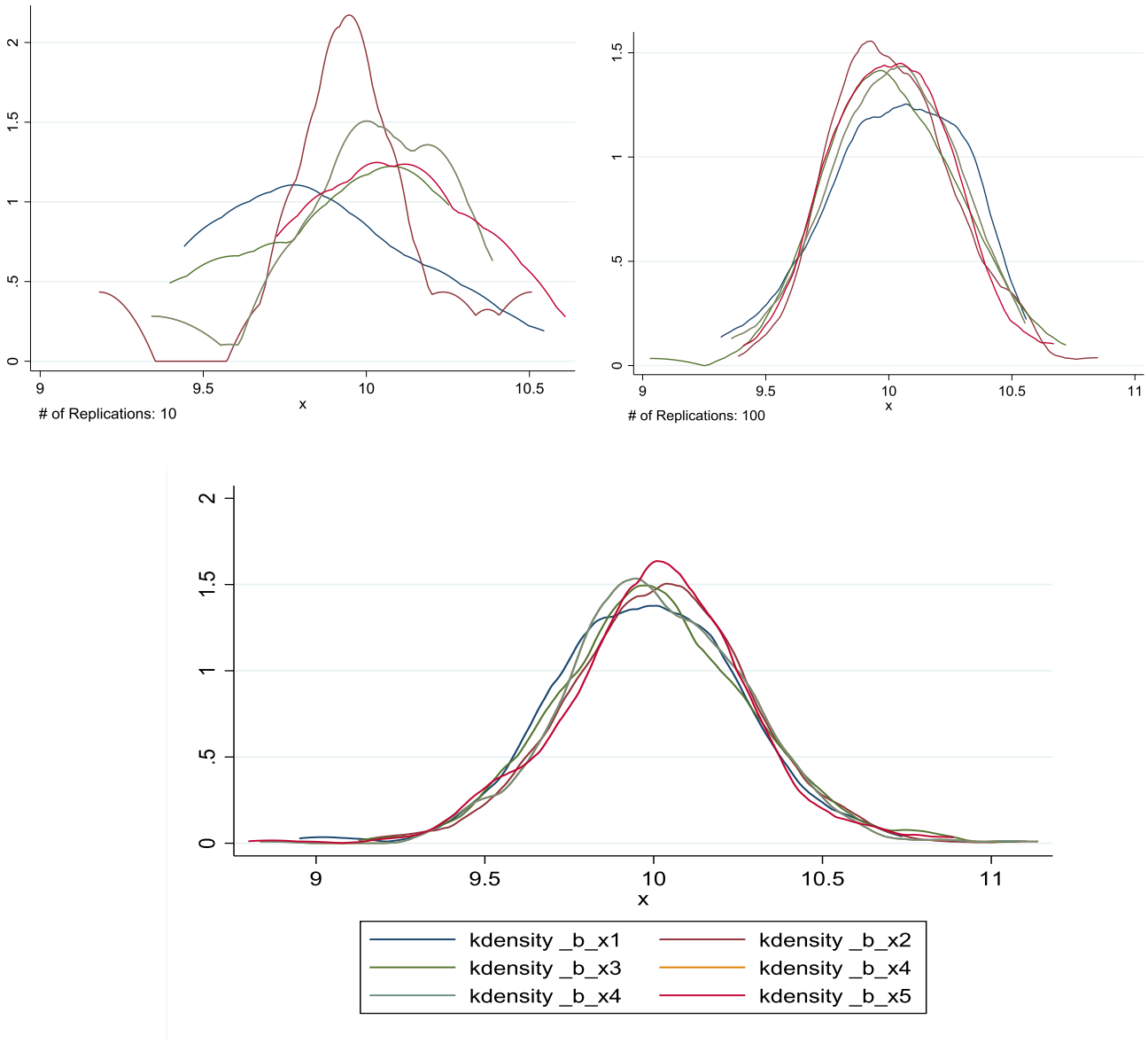
Variable	Replications	Mean	Std.Dev.	Min	Max
_b_x1	500	9.937	.575	5.552	11.942
_b_x2	500	9.994	.599	7.56	12.789
_b_x3	500	9.957	.688	2.325	12.944
_b_x4	500	10.007	.583	8.089	14.52
_b_x5	500	9.999	.582	5.572	12.208
_b_cons	500	10.002	.535	7.253	12.053

Source: Own elaboration

We can see that the minimum and maximum values for the 500 hundred replications with n=10 tends to be more stable than when n=6 which is the micronumerosity simulation. In this case the mean values are also more accurate in terms to approach to the data generating process of equation (1).

Now considering the number of observations to 20, the pattern of the distributions for each parameter is shown ahead, indicating a possibly significant difference from the n=10 exercise because of the shape of the curves for each distribution are different.

Graph 3 OLS - Distributions of the Coefficients with n=20



The range of the distribution is somewhat more accurate (from 9 to 11 in the x axis) for all replications with the 20 observations, this tends to indicate that the precision of the estimates is increasing as expected. However, the shape of the curve is somewhat different but stills relies over 10. Which is a sign of the consistency and unbiasedness property of the estimator. The descriptive statistics from the 500-replication exercise within this number of observations reflects a good precision of the estimators.

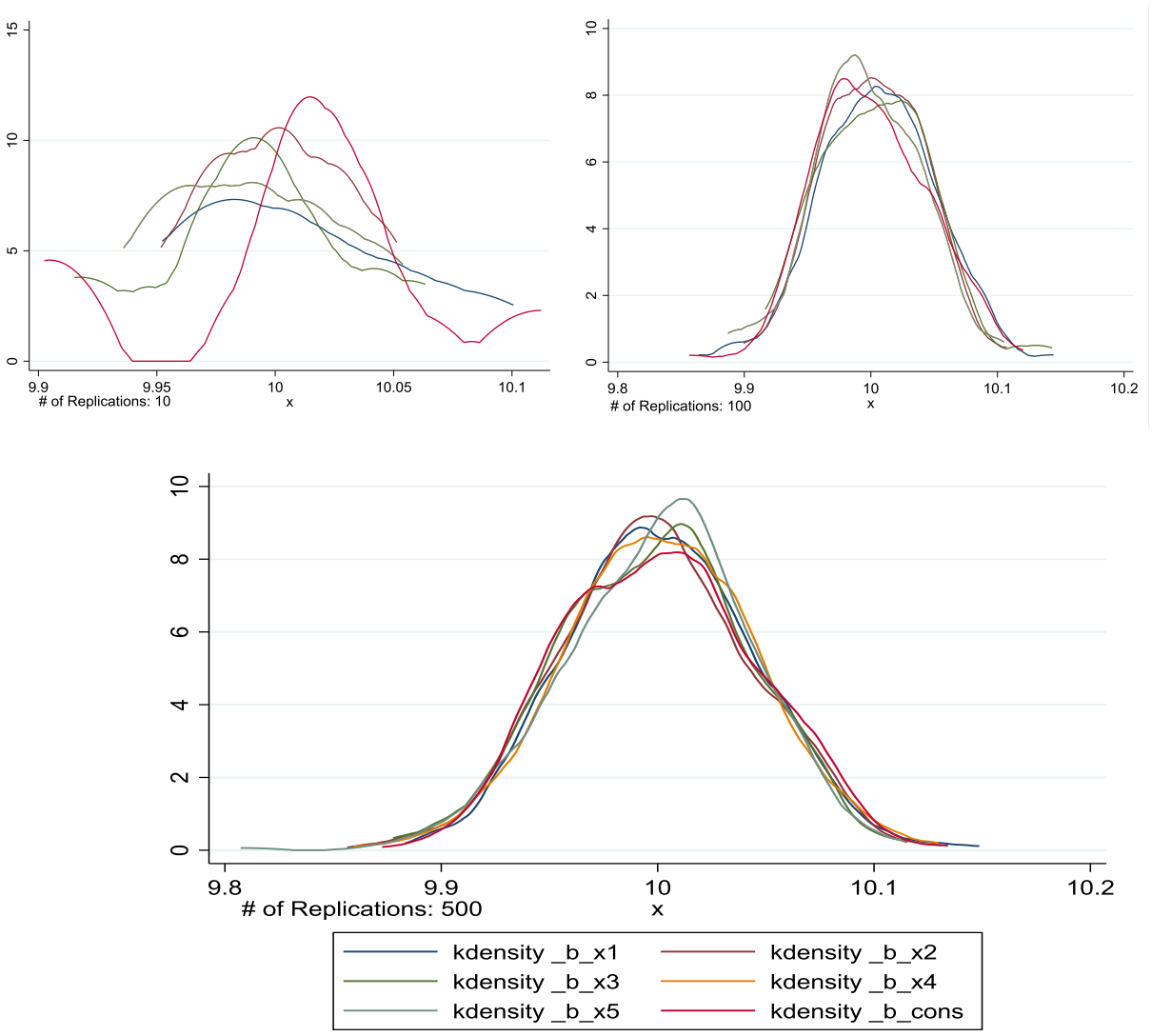
Table 8 OLS Descriptive Statistics with n=20

Variable	Replications	Mean	Std.Dev.	Min	Max
_b_x1	500	9.979	.274	8.95	10.736
_b_x2	500	10.018	.269	9.145	11.104
_b_x3	500	10.003	.284	9.126	11.082
_b_x4	500	10.007	.268	8.834	11.139
_b_x5	500	10.002	.275	8.8	10.889
_b_cons	500	9.998	.271	9.262	10.837

Source: Own elaboration

Finally, as a comparing exercise, we're setting the number of observations to 500 in order to understand the behavior of the coefficients' distribution.

Graph 4 OLS - Distributions of the Coefficients with n=500



Source: Own elaboration

As expected, the higher number of observations tend to have a faster converging rate to the true value of the parameters than the other simulations with lesser observations, the accuracy of the regressions are shown in the descriptive statistics ahead.

Table 9 OLS Descriptive Statistics n= 500

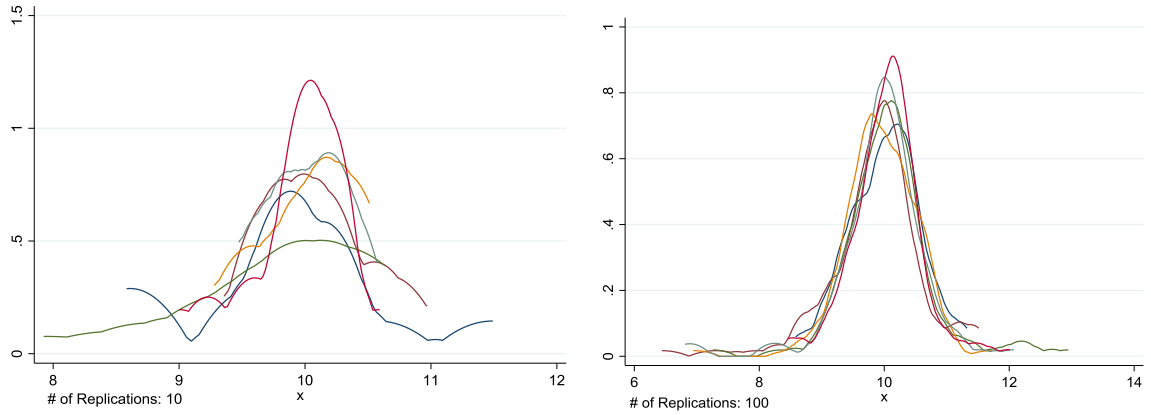
Variable	Replications	Mean	Std.Dev.	Min	Max
_b_x1	500	10.002	.043	9.883	10.149
_b_x2	500	9.999	.044	9.857	10.112
_b_x3	500	9.998	.043	9.878	10.115
_b_x4	500	10.001	.044	9.86	10.13
_b_x5	500	10.001	.044	9.807	10.132
_b_cons	500	10.001	.044	9.873	10.134

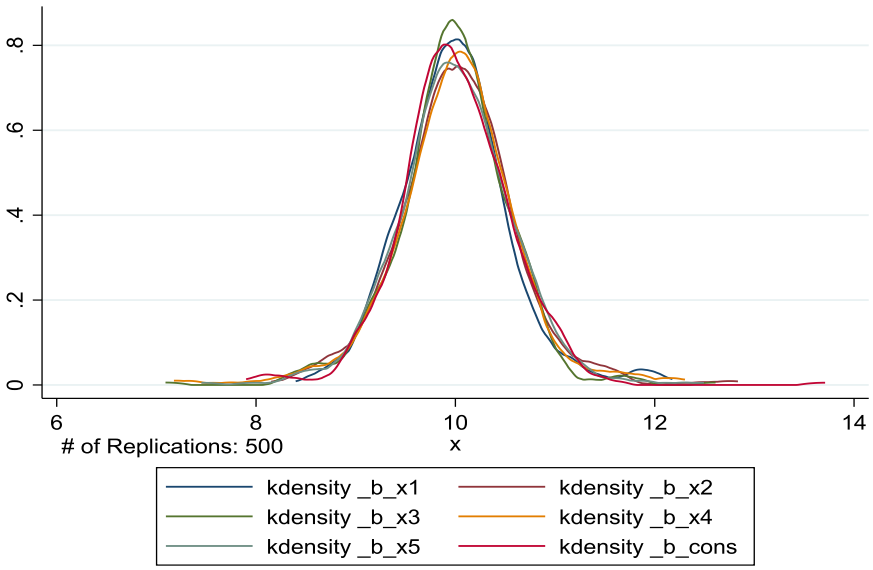
Source: Own elaboration

Jackknife

This type of estimation cannot be performed in the presence of perfect micronumerosity so distribution analysis cannot be done with the case of 6 observations. Moving ahead with 10 observations, the behavior of the distributions of the parameters according to different replications are shown in the next graphs.

Graph 5 Jackknife - Distributions of the Coefficients with n=10





Source: Own elaboration

It appears that the range of the different parameters' distributions in the case of 100 replications is higher than the rest of the simulations considering 10 observations, something particular but yet over the long-run not important since the mean value of all replications stills converge to the true value. The shape of the distributions cannot be established as better from the OLS, since the range varies widely. From this, descriptive statistics would be useful.

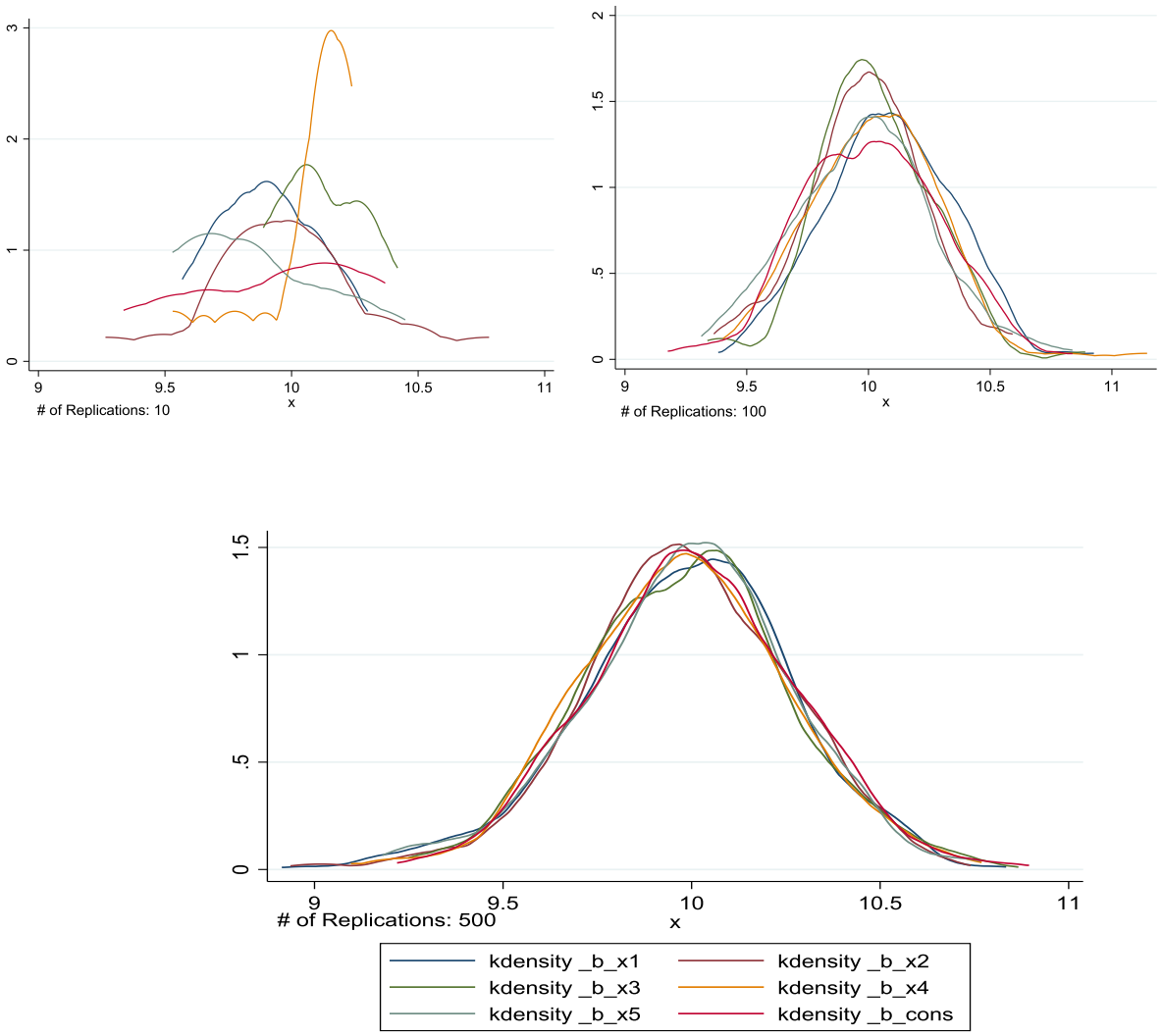
Table 10 Jackknife Descriptive Statistics n=10

Variable	Replications	Mean	Std.Dev.	Min	Max
_b_x1	500	9.999	.554	8.398	12.178
_b_x2	500	10.024	.594	8.152	12.835
_b_x3	500	9.987	.554	7.092	12.61
_b_x4	500	10.009	.63	7.176	12.303
_b_x5	500	10.001	.577	7.451	12.494
_b_cons	500	9.997	.565	7.9	13.709

Source: Own elaboration

The expected value of the parameters is more accurate in the jackknife simulations than it is with the OLS, also the standard deviation tends to be lower for the jackknife approach. Considering now a sample size of 20 observations, the following pattern can be observed.

Graph 6 Jackknife - Distributions of the Coefficients with n=20



Source: Own elaboration

Jackknife estimation seems to be more unstable with a lower number of replications considering n=20, however we're not sure yet if it's suitable better than OLS by the graphic interpretation, looking at the descriptive statistics

Table 11 Jackknife Descriptive Statistics n=20

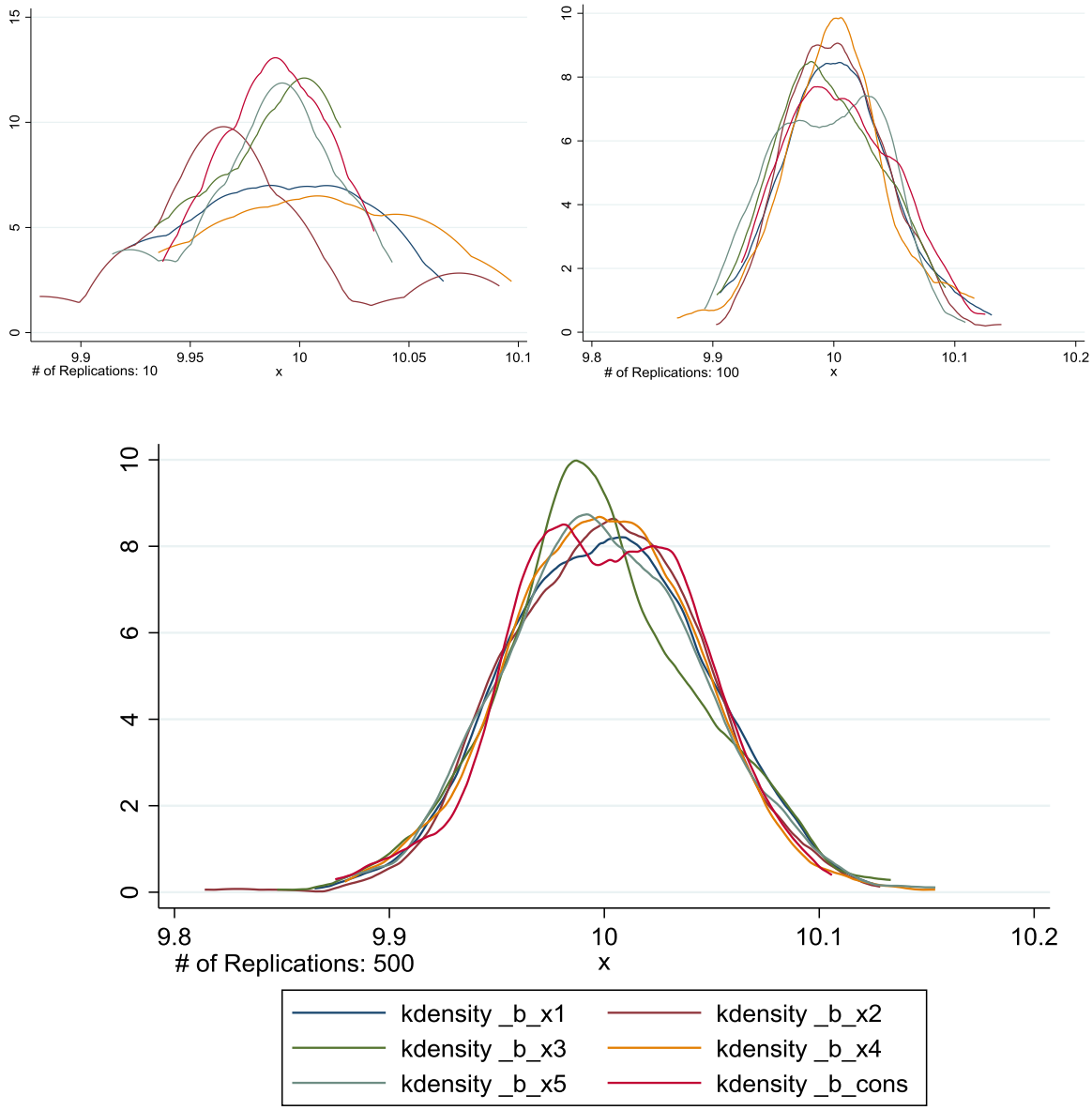
Variable	Replications	Mean	Std.Dev.	Min	Max
_b_x1	500	9.985	.284	8.913	10.834
_b_x2	500	9.986	.272	8.936	10.74
_b_x3	500	9.988	.273	9.249	10.867
_b_x4	500	9.978	.274	9.097	10.769
_b_x5	500	9.99	.276	9.181	10.766
_b_cons	500	10.006	.274	9.219	10.895

Source: Own elaboration

The estimations with jackknife seem to be pretty close to the ones performed with OLS at this number of observations, however OLS seems to have the advantage to be more stable with lesser replications than Jackknife does and the expected value with n=20 of the estimators is closer to the DGP than it is for jackknife.

Finally, with 500 observations, it is noted that jackknife has the counterpart to require a higher and significant time of computing during the estimations.

Graph 7 Jackknife - Distributions of the Coefficients with n=500



Source: Own elaboration

The jackknife distribution with n=500 seems to converge somewhat equal to the OLS estimations. If we analyze the statistics relative to the OLS for the same number of observations we'll find that the OLS performs better in terms of the standard deviation and minimum and maximum values closer to 10.

Table 12 Jackknife Descriptive Statistics n=500

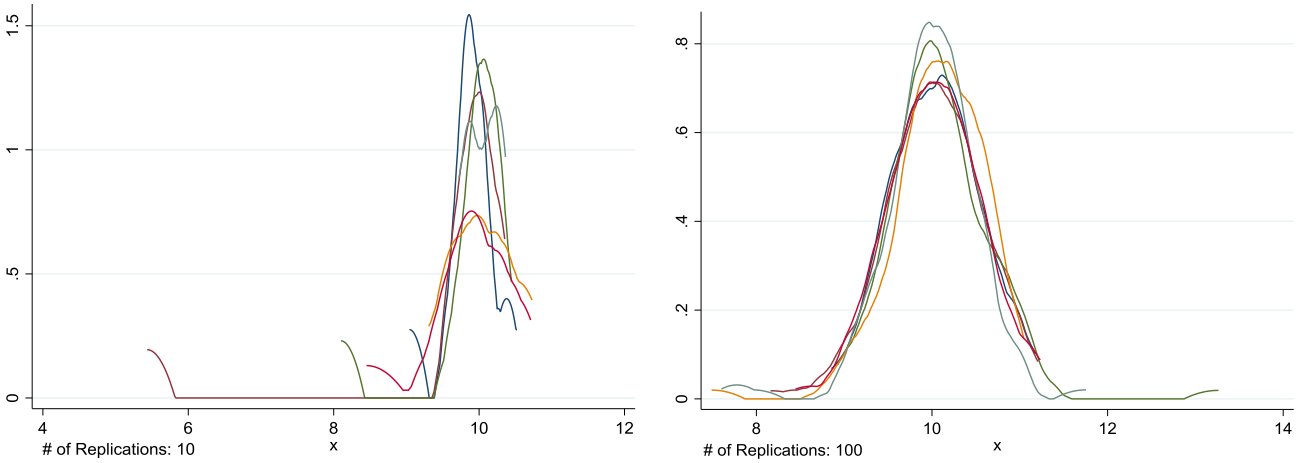
Variable	Obs	Mean	Std.Dev.	Min	Max
_b_x1	500	10.002	.045	9.865	10.126
_b_x2	500	10.001	.044	9.814	10.128
_b_x3	500	9.999	.046	9.848	10.133
_b_x4	500	10	.043	9.879	10.154
_b_x5	500	10	.045	9.875	10.154
_b_cons	500	10	.043	9.875	10.106

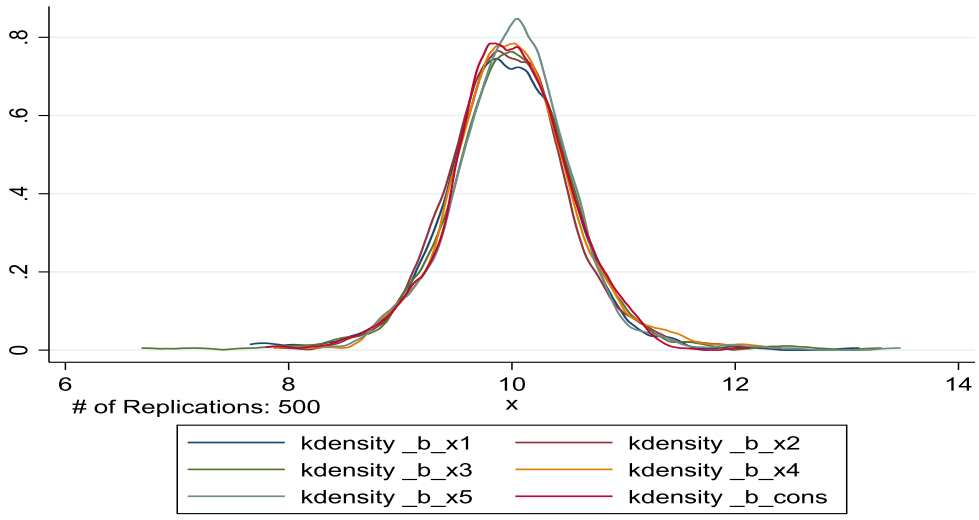
Source: Own elaboration

Bootstrap

Similar to the Jackknife approach, bootstrap estimation cannot be performed if the number of observations is 6, so it is not allowed perfect micronumerosity. Moving to the analysis with n=10 we can observe the following patterns of the parameters via bootstrap.

Graph 8 Bootstrap - Distributions of the Coefficients with n=10





Source: Own elaboration

The pattern related to the lowest replications (10) tends to be unstable with the bootstrap technique with n=10, but as it gets more replications the parameters converge to their true value. The descriptive statistics are presented ahead indicating a similar behavior to the Jackknife technique.

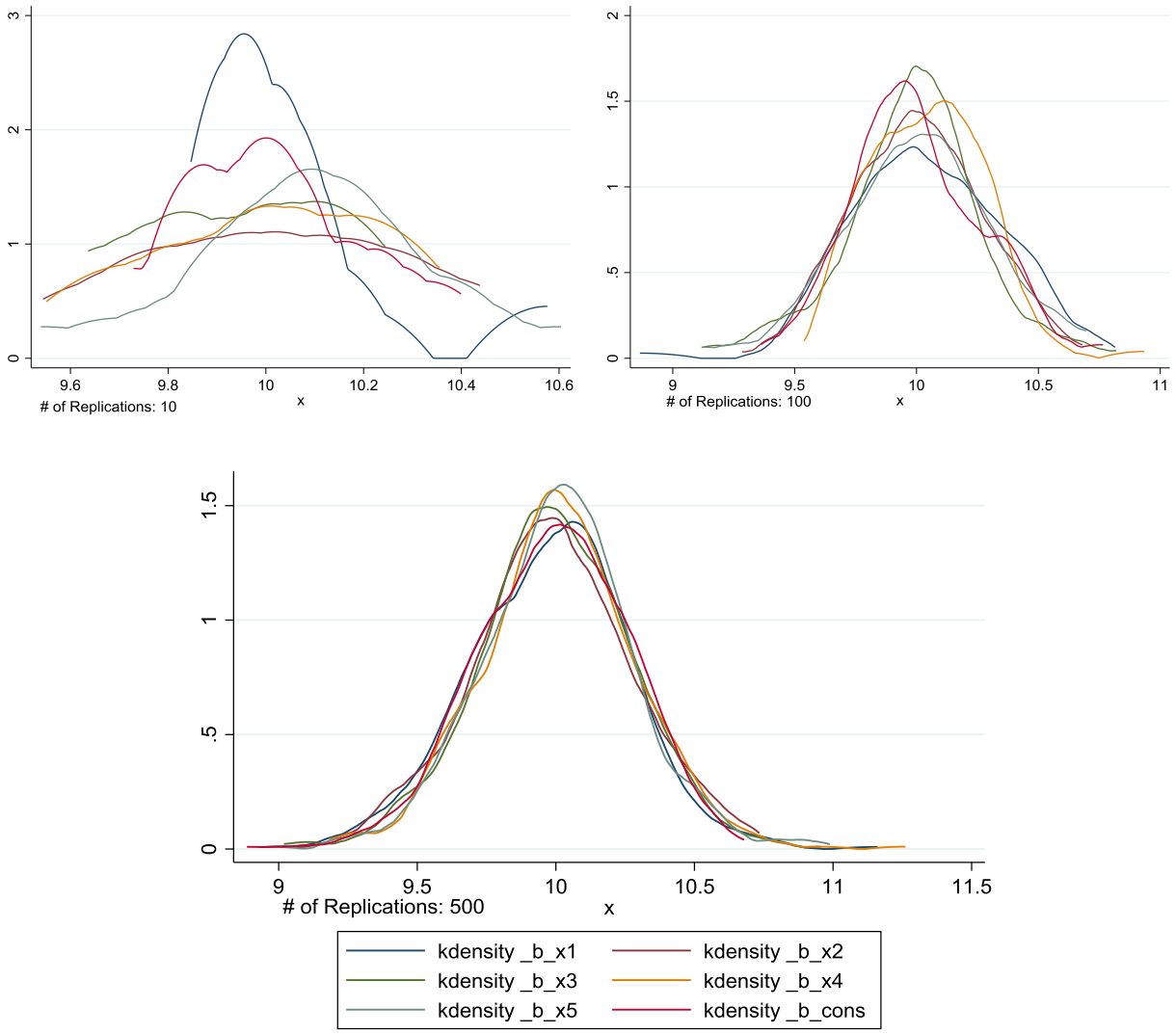
Table 13 Bootstrap Descriptive Statistics n=10

Variable	Replications	Mean	Std.Dev.	Min	Max
_b_x1	500	9.955	.588	7.653	13.115
_b_x2	500	9.968	.552	7.915	12.409
_b_x3	500	9.995	.633	6.682	13.314
_b_x4	500	10.014	.558	7.865	12.28
_b_x5	500	10.015	.556	7.969	13.482
_b_cons	500	9.98	.532	7.794	12.132

Source: Own elaboration

Moving to n=20, the patterns relative to the lesser replications tend to be more stable than with n=10, indicating a sensitive behavior of the bootstrap with lower samples, however stills yielding results similar to OLS and Jackknife.

Graph 9 Bootstrap - Distributions of the Coefficients with n=20



Source: Own elaboration

As happens with the jackknife, the bootstrap seems to have variations for each distribution of each variable when the replication number is set to 100, however the distributions converge as OLS and jackknife in the case of bootstrap when replications are set to 500.

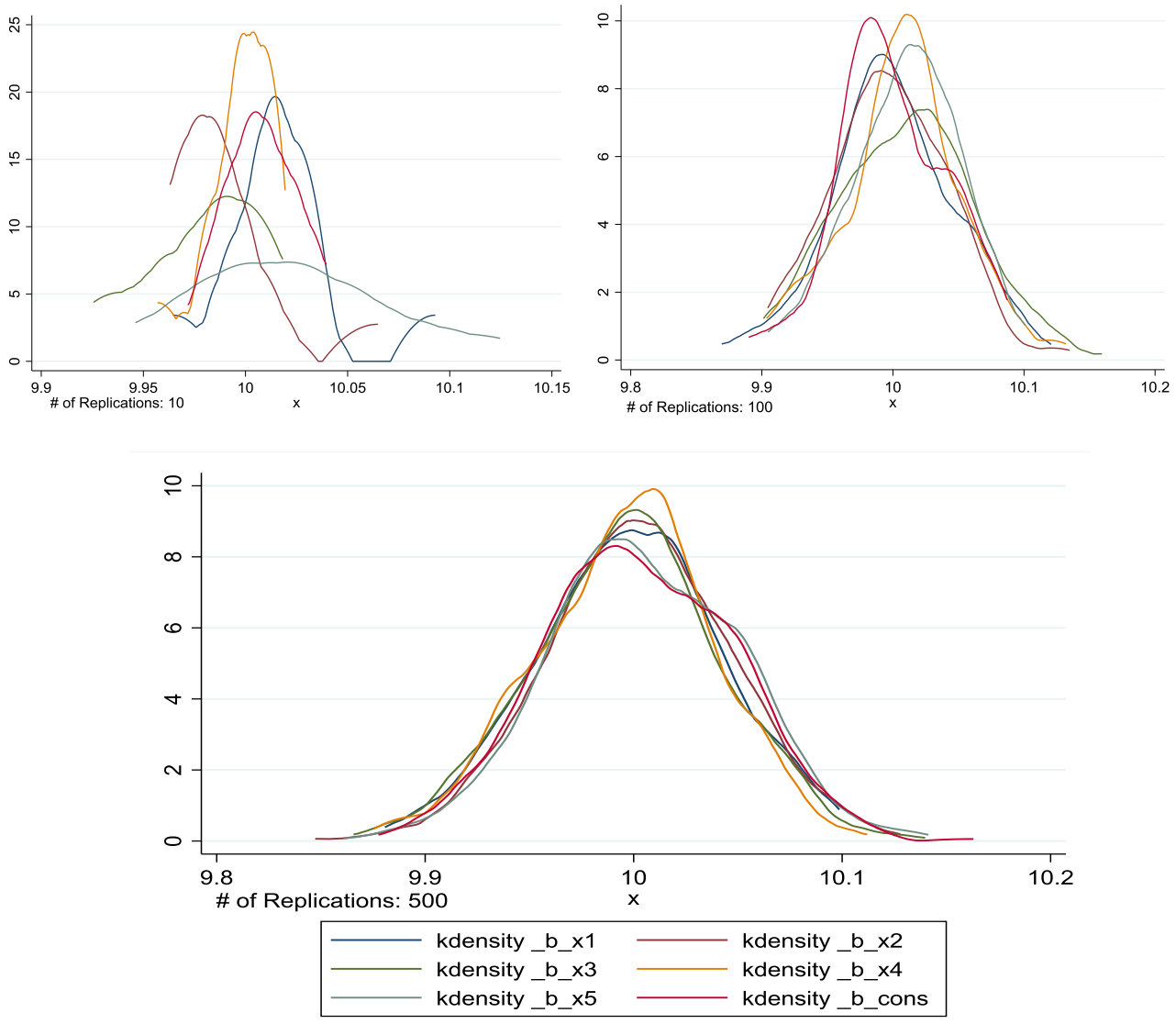
Table 14 Bootstrap Descriptive Statistics n=20

Variable	Obs	Mean	Std.Dev.	Min	Max
_b_x1	500	9.977	.288	8.93	11.161
_b_x2	500	9.986	.288	9.015	10.734
_b_x3	500	9.998	.277	9.019	10.957
_b_x4	500	10.013	.279	9.185	11.261
_b_x5	500	10.009	.275	8.953	10.988
_b_cons	500	9.988	.273	8.885	10.678

Source: Own elaboration

Going further with the bootstrap technique and using n=500 observations, the graphical pattern indicates some better adjustment regarding to lower replications.

Graph 10 Bootstrap - Distributions of the Coefficients with n=500



Source: Own elaboration

The pattern of the distributions among the coefficients when the number of replications is set to 500 tends to be more different from the OLS and the Jackknife estimations, which might suggest that bootstrap performs different distributions for each estimator even when the OLS and jackknife tend to converge the distribution for all estimators with the same number of n=500 observations. According to the descriptive statistics, bootstraps seems to be as efficient as OLS and Jackknife

specially because of the mean value of the coefficients, it's stills as accurate relative to this expected value in comparison.

Table 15 Bootstrap Descriptive Statistics n=500

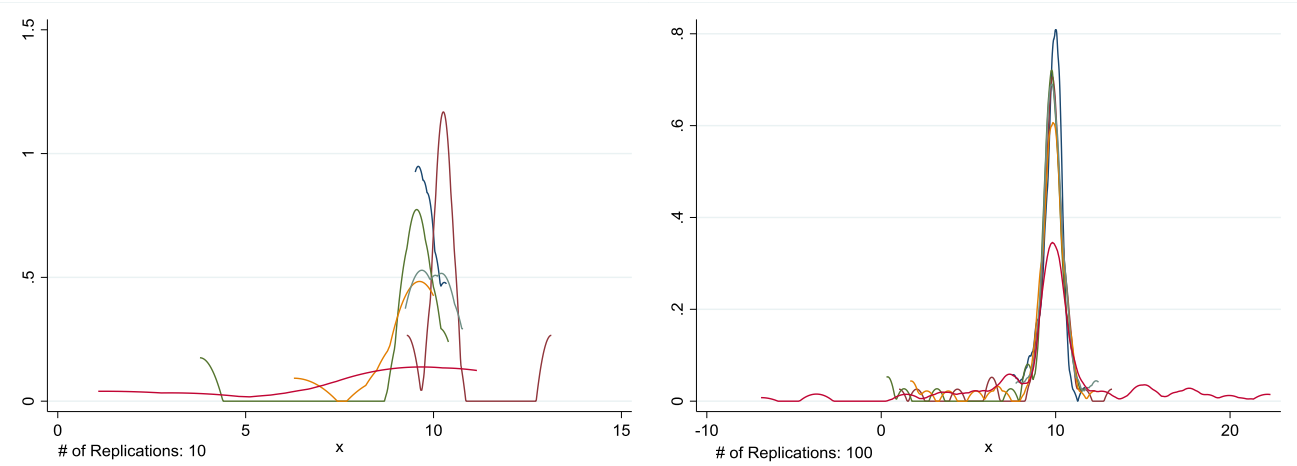
Variable	Obs	Mean	Std.Dev.	Min	Max
_b_x1	500	9.998	.044	9.881	10.099
_b_x2	500	10.003	.044	9.847	10.128
_b_x3	500	9.997	.045	9.866	10.14
_b_x4	500	9.996	.042	9.875	10.112
_b_x5	500	10.005	.045	9.861	10.141
_b_cons	500	10.002	.045	9.878	10.163

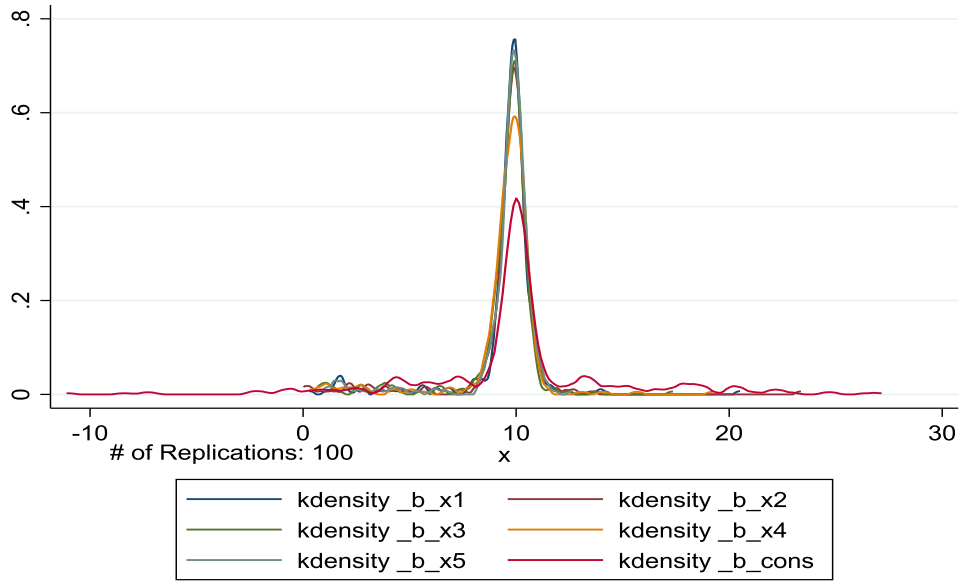
Source: Own elaboration

Lasso regression

As mentioned before, lasso cannot compute the model when the number of observations is equal to 6, so we're going straight to the analysis with 10 observations, the graphical pattern is shown ahead.

Graph 11 Lasso - Distributions of the Coefficients with n=10





Source: Own elaboration

The graphs suggest that the distributions are different for each variable across replications, in that case the constant coefficient remains with difference ranges when its converging to the true parameter. The descriptive statistics suggest that from the 500 simulations some of them failed and were just covering up to 307 replications, the constant term was the only which remained across regressions however even when the mean value it's somewhat accurate, the minimum and maximum values are varying more than the coefficients associated with x variables.

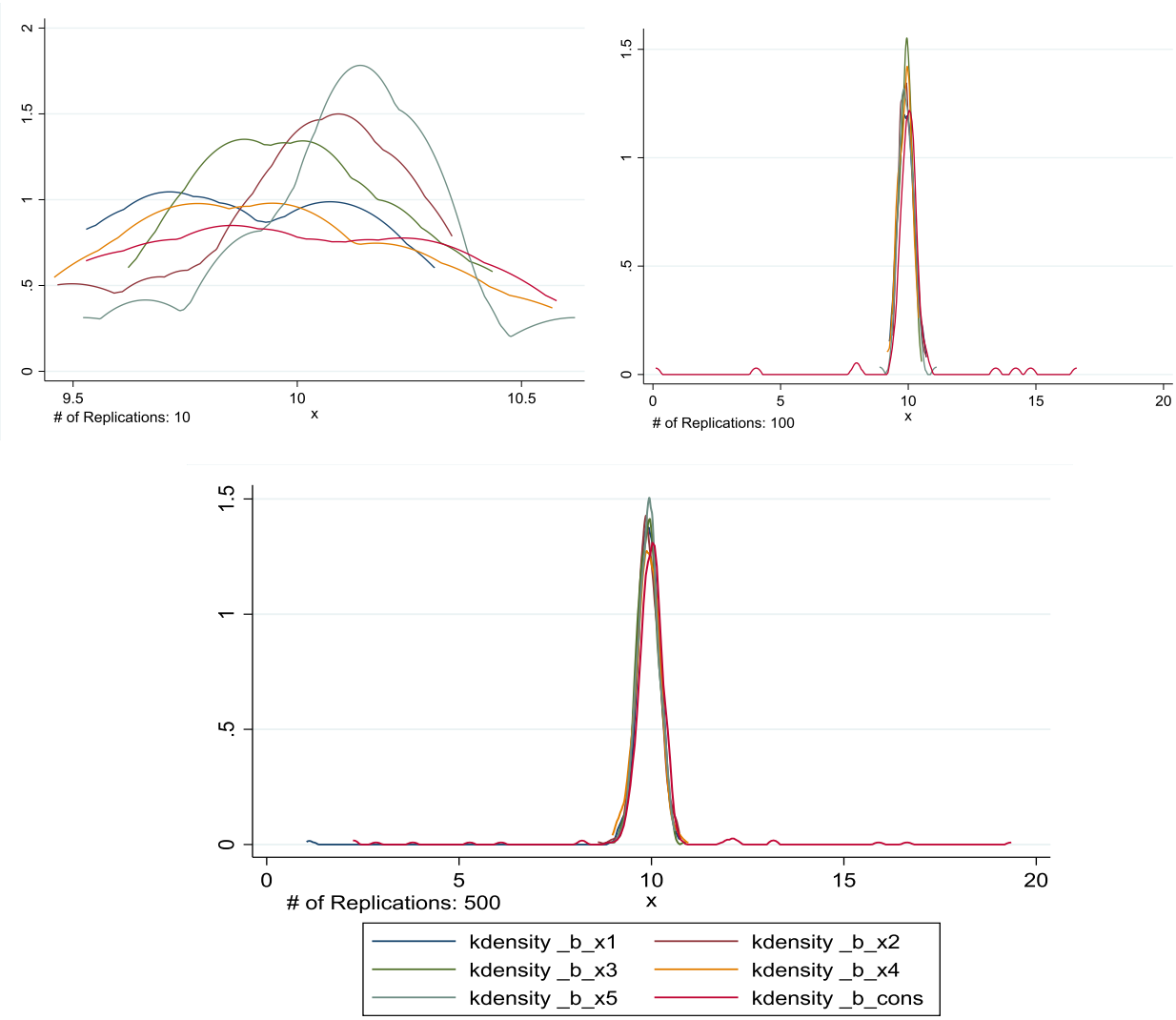
Table 16 Lasso Descriptive Statistics n=10

Variable	Obs	Mean	Std.Dev.	Min	Max
_b_x1	317	9.484	1.963	.287	20.51
_b_x2	317	9.444	2.142	.014	23.37
_b_x3	314	9.371	1.921	.657	17.363
_b_x4	318	9.382	2.063	.29	19.233
_b_x5	307	9.457	1.841	.894	12.664
_b_cons	500	10.108	4.509	-11.102	27.153

Source: Own elaboration

Proceeding with lasso estimations with n=20 we watch the graphical pattern associated to the distribution of the parameters as it follows:

Graph 12 Lasso - Distributions of the Coefficients with n=20



Source: Own elaboration

According to the distributions, the estimators associated to the different variables seem to behave over a wide range during the simulations with n=20 observations. Relying to the descriptive statistics we can find a significant range regarding to x_1 variable and the constant term in the regression. Also, some simulations failed to accomplish the main total of 500, which tends to indicate that lasso approach is sensitive to the number of replications and the overall range of the estimators across replications.

Table 17 Lasso Descriptive Statistics n=20

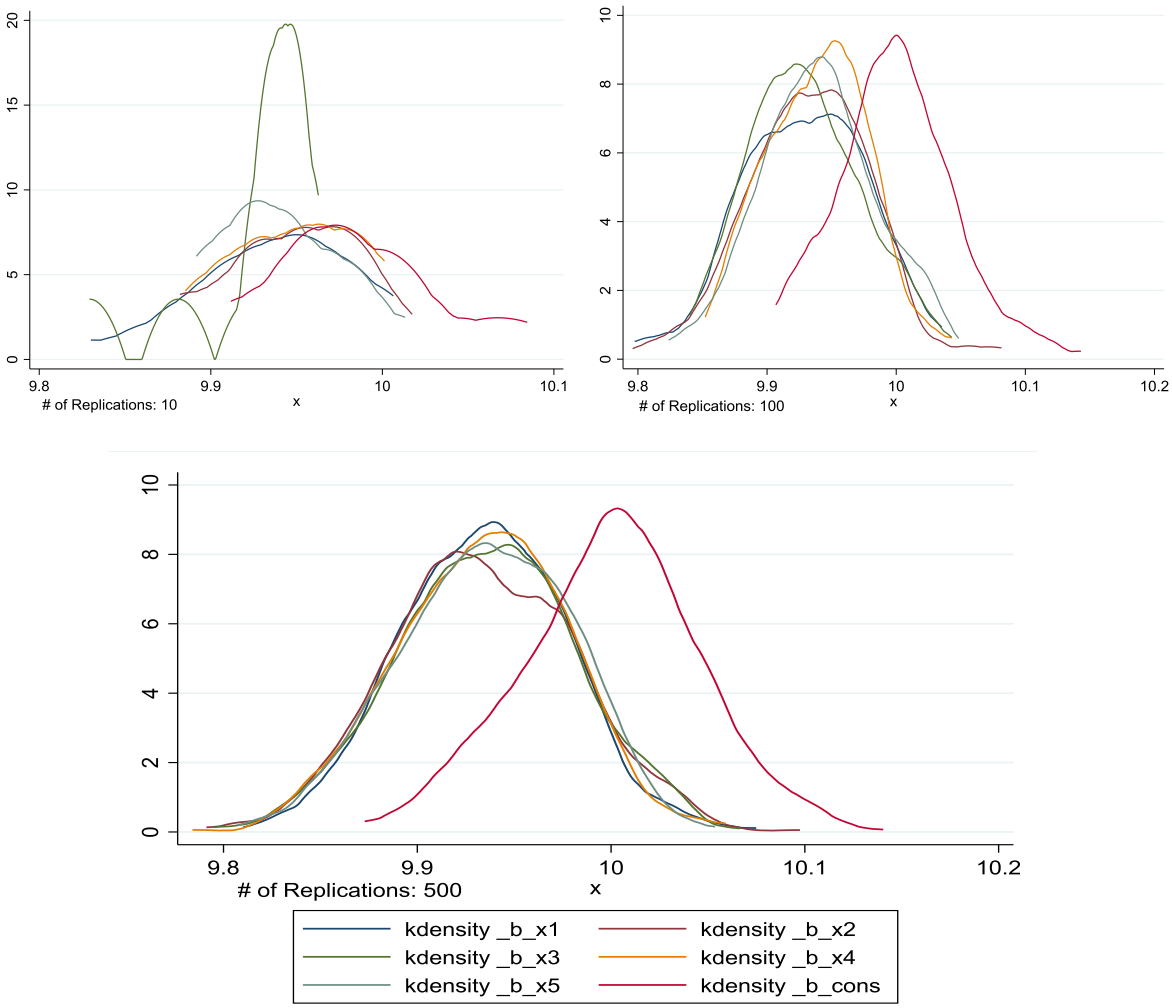
Variable	Obs	Mean	Std.Dev.	Min	Max
_b_x1	476	9.889	.635	1.037	10.743
_b_x2	474	9.908	.296	8.608	10.746
_b_x3	474	9.904	.276	8.965	10.901

_b_x4	474	9.903	.31	8.986	10.962
_b_x5	474	9.917	.272	8.957	10.609
_b_cons	500	9.98	1.002	2.243	19.351

Source: Own elaboration

The descriptive statistics tend to indicate some instability of the lasso regression with n=10 and 20, which would be judge in overall with the 500 observations simulations. Proceeding with the analysis with n=500 simulations, the graphical pattern is shown ahead.

Graph 13 Lasso - Distributions of the Coefficients with n=500



Source: Own elaboration

The distribution seems not to converge to the exact value of the DGP, lasso regression also seems to perform a different distribution relative to the other x variables and the constant coefficient. This doesn't mean Lasso regression is inconsistent, since it's close to 10, however is not as consistent as other estimations are. The descriptive statistics of the estimated parameters tends to

confirm this idea since the expected value of the estimators is not as close to the other types of estimations, also it tends to have a standard deviation a little bit higher than the others.

Table 18 Lasso Descriptive Statistics n=500

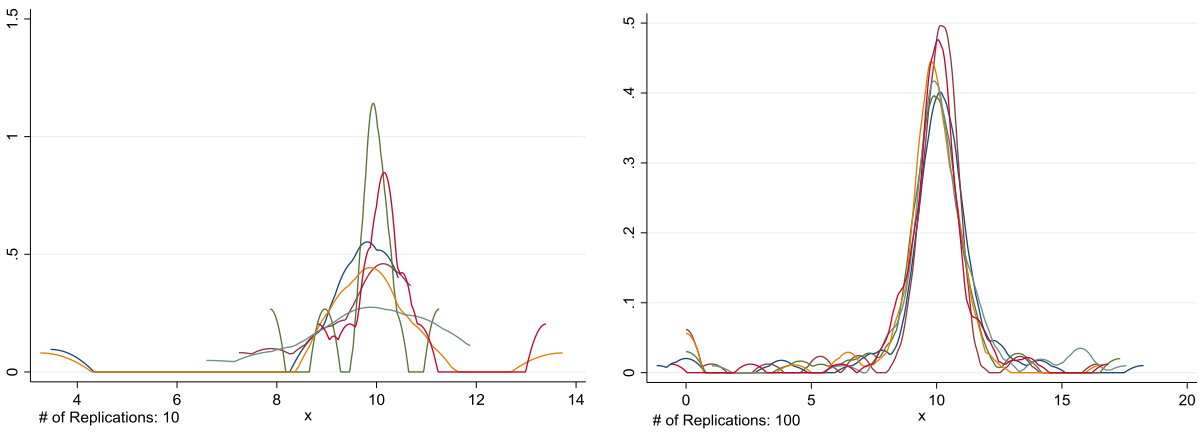
Variable	Obs	Mean	Std.Dev.	Min	Max
_b_x1	500	9.934	.043	9.81	10.075
_b_x2	500	9.934	.047	9.791	10.097
_b_x3	500	9.936	.046	9.795	10.067
_b_x4	500	9.934	.044	9.784	10.059
_b_x5	500	9.935	.044	9.807	10.054
_b_cons	500	9.999	.046	9.873	10.14

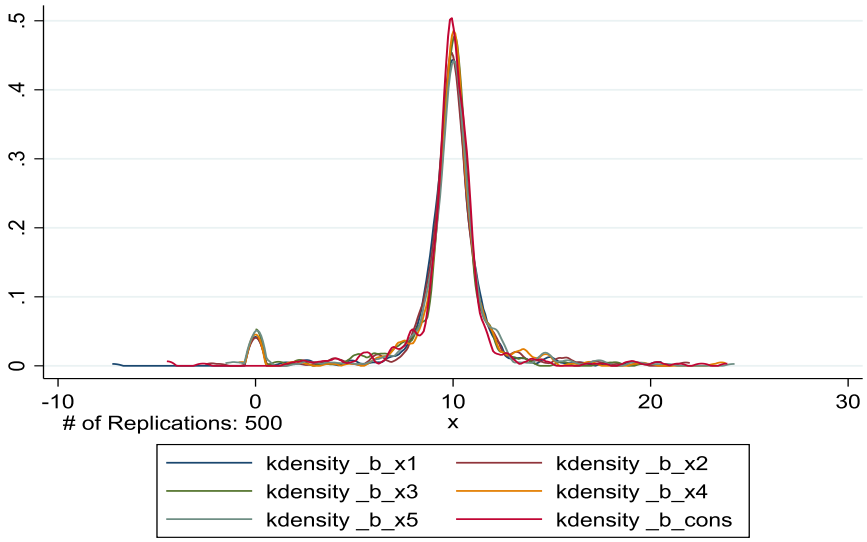
Source: Own elaboration

Robust regression

The last type of estimation we're analyzing is the robust regression, this one cannot be estimated with n=6 observations (the perfect micronumerosity case) so we're going straight forward to set n=10 observations and perform the graphical distribution patterns.

Graph 14 Robust Regression - Distributions of the Coefficients with n=10





Source: Own elaboration

With 500 simulations, Stata calculated 484 replications, the rest of the remaining replications failed in the maximization process. There are some appoints to make here, first: the range of the distribution with n=10 observations across replications is way too high in comparison OLS, Jackknife, Bootstrap or Lasso types of estimations, second: some of the distributions of some variables tend to have spikes closer to the value of 0 indicating that a significant number of times, the robust regression adjusted some coefficients as 0. According to the descriptive statistics, the mean value of the coefficients tends to converge better than Lasso, however Jackknife and Bootstrap perform better with this set of observations.

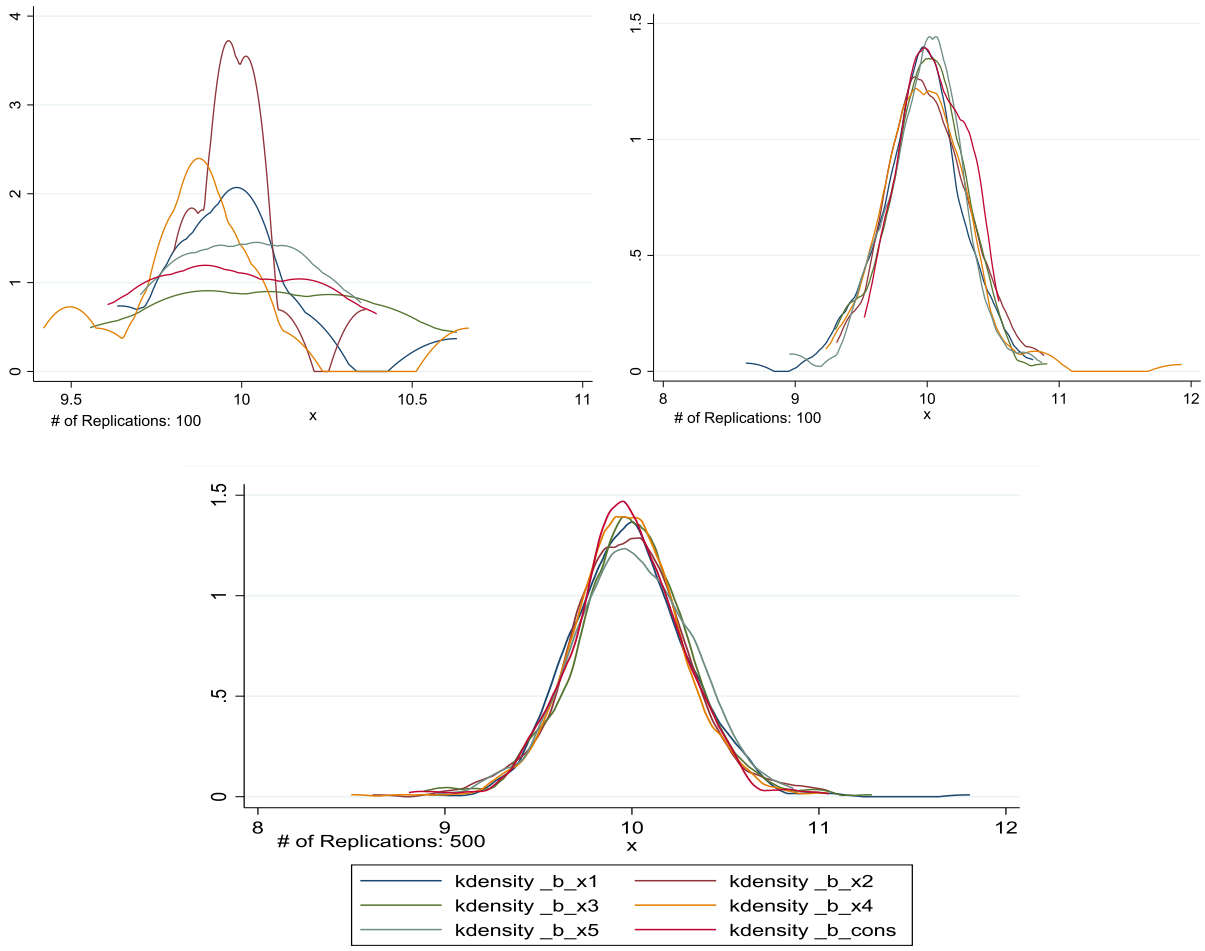
Table 19 Robust Regression Descriptive Statistics n=10

Variable	Obs	Mean	Std.Dev.	Min	Max
_b_x1	484	9.704	2.682	-7.252	20.553
_b_x2	484	9.732	2.689	-2.542	21.981
_b_x3	484	9.494	2.57	0	19.703
_b_x4	484	9.794	2.661	-.849	23.795
_b_x5	484	9.694	2.949	-1.525	24.251
_b_cons	484	9.898	2.408	-4.489	23.887

Source: Own elaboration

Moving forward and setting n=20 observations, we can observe that the graphical pattern of the distributions for each estimator of each variable is going more accurate with the robust regression technique, however no significant changes are from the other types of estimations.

Graph 15 Robust Regression - Distributions of the Coefficients with n=20



Source: Own elaboration

The behavior with n=20 observations is far better than with n=10, also these results are consistent with a lesser range over the estimators. The mean value of the estimators is getting closer to 10 as we increased the number of replications.

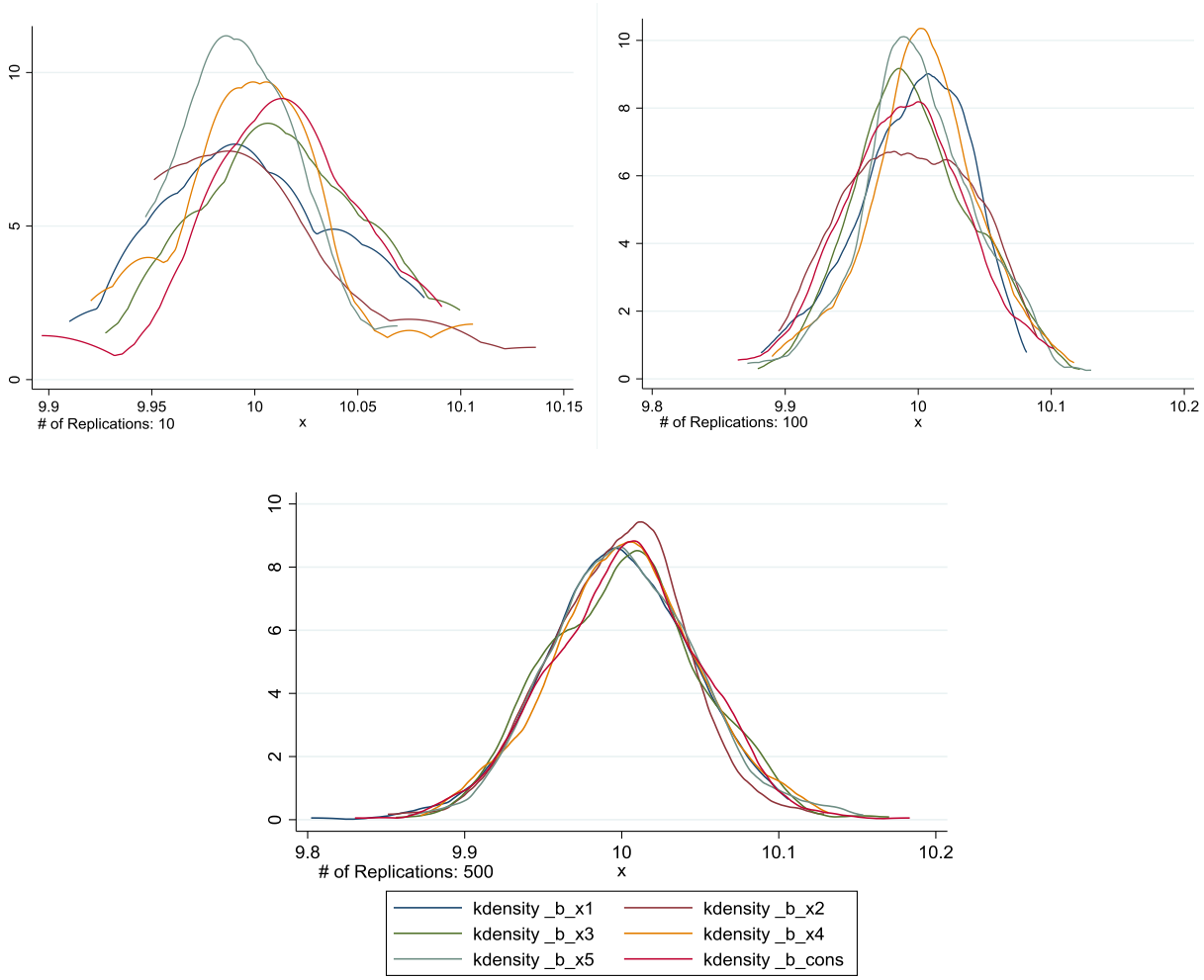
Table 20 Robust Regression Descriptive Statistics n=20

Variable	Obs	Mean	Std.Dev.	Min	Max
_b_x1	500	9.989	.313	8.915	11.808
_b_x2	500	9.981	.324	8.613	11.054
_b_x3	500	9.998	.32	8.888	11.283
_b_x4	500	9.974	.3	8.499	11.044
_b_x5	500	10.003	.317	8.958	10.864
_b_cons	500	9.972	.3	8.808	11.054

Source: Own elaboration

Setting the final simulations with n=500, the results of the distributions with kernel densities are shown ahead.

Graph 16 Robust Regression - Distributions of the Coefficients with n=500



Source: Own elaboration

The patterns tend to indicate a convergence to the true value of the parameter as the number of replications are increased, also with the descriptive statistics the mean value is closer to 10, leading to think that robust regression is also a good option in large samples.

Table 21 Robust Regression Descriptive Statistics n=500

Variable	Obs	Mean	Std.Dev.	Min	Max
_b_x1	500	9.998	.046	9.802	10.106
_b_x2	500	9.997	.043	9.851	10.129

_b_x3	500	10.002	.048	9.856	10.17
_b_x4	500	10.002	.046	9.872	10.132
_b_x5	500	10.002	.047	9.864	10.154
_b_cons	500	10.002	.048	9.83	10.183

Source: Own elaboration

Comparing the estimations

In order to synthetize the results of the previous part we can discriminate by the number of observations (the lowest) and the descriptive statistics for the coefficients, in this order of ideas the mean value of the whole estimators across simulations would be our reference point, standard deviation as lower it is the better, and the minimum and maximum values closer to 10 would be ranked.

Table 22 Comparison between Estimations, n=10

Estimation Type n=10	Expected Value of the Estimators	Expected Std. Deviation	Expected Min	Expected Max
OLS	9,98266667	0,59366667	6,0585	12,7426667
Jackknife	10,0028333	0,579	7,69483333	12,6881667
Bootstrap	9,98783333	0,56983333	7,64633333	12,7886667
Lasso	9,541	2,4065	-1,49333333	20,0488333
Robust Regression	9,71933333	2,65983333	-2,77616667	22,3616667
Best Option	Jackknife	Bootstrap	Jackknife	Jackknife

Source: Own elaboration

When we're considering a set of sample size with n=10 observations in the context of a 6-coefficient estimation model, the best option is the jackknife estimation technique. It should be noted that the number of freedom degrees in the residuals for this case is equal to 4. It is expected that when this number gets higher, we might have more accurate estimators from the other techniques.

Table 23 Comparison between Estimations, n=20

Estimation Type n=20	Expected Value of the Estimators	Expected Std. Deviation	Expected Min	Expected Max
OLS	10,0011667	0,2735	9,0195	10,9645
Jackknife	9,98883333	0,2755	9,09916667	10,8118333
Bootstrap	9,99516667	0,28	8,99783333	10,9631667
Lasso	9,91683333	0,46516667	6,466	12,2186667
Robust Regression	9,98616667	0,31233333	8,78016667	11,1845
Best Option	OLS	OLS/Jackknife	Jackknife	Bootstrap

Source: Own elaboration

When the number of observations is increased to n=20 and the degrees of freedom are higher to a value of 14, the OLS performs quite better in the expected value of the coefficients, mean while we got a draw with OLS and jackknife in the case for the minimum expected value of the standard deviation.

Table 24 Comparison between Estimations, n=500

Estimation Type n=500	Expected Value of the Estimators	Expected Std. Deviation	Expected Min	Expected Max
OLS	10,0003333	0,04366667	9,85966667	10,1286667
Jackknife	10,0003333	0,04433333	9,85933333	10,1335
Bootstrap	10,0001667	0,04416667	9,868	10,1305
Lasso	9,94533333	0,045	9,81	10,082
Robust Regression	10,0005	0,04633333	9,84583333	10,1456667
Best Option	Bootstrap	OLS	Bootstrap	Lasso

Source: Own elaboration

Finally, when our sample size is sufficiently large (n=500) the bootstrap technique performs better than OLS, Jackknife, Lasso or Robust regression however OLS tends to have a lesser expected deviation than the rest. Over this stage since samples are large, there are sufficient arguments to prefer one method over other, for example, robust regression wasn't scored as the best in any of these statistics but it would be extremely useful when we got outliers or such things.

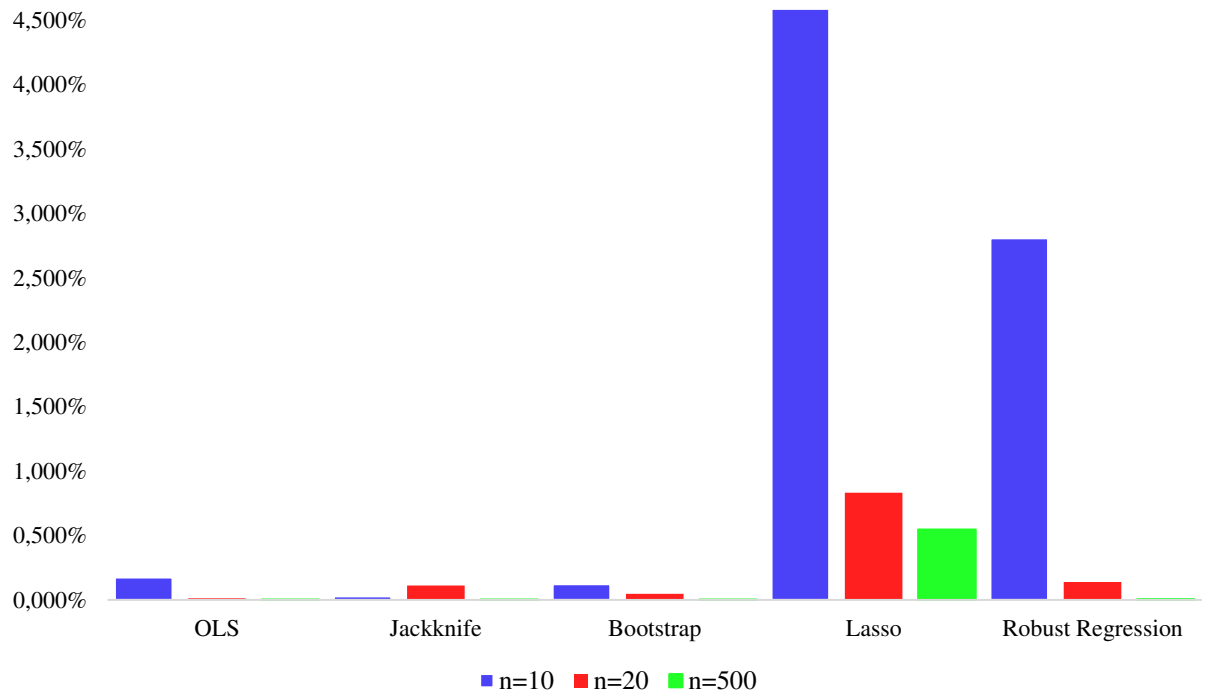
We need to remember that this analysis was performed with random variables which followed $N\sim(0,1)$ and the main interest was to analyze the estimations for low samples (the perfect micronumerosity case n=6 and the others with n=10, n=20 with 500 hundred simulations). And also, the DGP in equation (1) was also established to be a cross-sectional type of data, so no autoregressive problems or incorrect specifications were tested for the estimation's types.

The relative bias analysis with n=10 observations, suggest that Lasso regression performs the worse bias value, reaching a 4.59% calculated from the expected value of the estimators in reference with the true parameter, followed by the robust regression with a value of 2.807%. Bootstrap and OLS performs far better than this types with respective scores of 0.173% and 0.122%, the lower bias was obtained with the jackknife approach with a bias of 0.028%.

Moving to the sample size of n=20 observations, Lasso and robust regression performs also the worse value of the relative bias, respectively with values of 0.832% and 0.138%. Jackknife now turns to be in the third place with a relative bias of 0.112% while bootstrap has a value of 0.048% and the OLS with a score 0.012% indicating a lesser bias.

Finally, when the sample size is large (n=500), Lass remains with the worse score in the relative bias with a value of 0.547%, meanwhile robust regression has a score of 0.005% of relative bias against the DGP. OLS and Jackknife have the same relative bias with a score of 0.003%. The best performance in terms of relative bias in this case was obtained with Bootstrap with a score of 0.002% of relative bias among the simulations, the proceeding graph summarizes this result.

Graph 17 Relative Bias for each estimation type by sample size



Source: Own elaboration

Conclusions

This paper performed over 1500 simulations distributed among different sample sizes (n=6, n=10, n=20 and n=500) with a linear Data Generating Process in order to regress a model with six coefficients and five variables, these variables were randomly distributed with zero mean and variance of one, the estimations types for the regressions across simulations were the approaches of OLS, Jackknife, Bootstrap, Lasso and Robust Regression.

The statistical significance of the coefficients across the models tend to follow the pattern described by Speed (1994) where a significant relationship found in a small sample also will prevail when the sample size gets bigger. However, the Bootstrap approach seems to be sensitive to the sample size since with n=10 observations it didn't present a significant relation for one variable which was part of the DGP, suggesting that Bootstrap might discard a significant relation of certain variables with a small sample size. As soon as the sample size increased to n=20, the bootstrap approach presented significant relations with a 5% significance level, and with a larger sample size, the statistical significance was of 1%. On the other hand, OLS, Jackknife, Lasso and Robust regression performed well in terms of the statistical significance of the coefficients for all the variables in the DGP across the Monte Carlo simulations with different sample size.

Comparing the results with n=10 observations, the best estimation type was performed with the Jackknife approach, since the expected value of the coefficients was the best in terms to be closer to the true value of the DGP, also this approach suggests a lesser relative bias across the replications for the coefficients. Bootstrap on the other hand with this sample size had the lowest expected standard deviation. In this case, it is confirmed that Speed (1994) was right in affirming that Jackknife and Bootstrap techniques are more suitable in small samples, however the drawback of the bootstrap approach is the sensitive in the statistical significance of the coefficients. According to these results, the jackknife approach seems to be more suitable for lower sample analysis.

In the case of n=20 observations, OLS obtained the best score regarding the accuracy of the estimators across simulations, as a reference for this, the relative bias was the lowest among the other estimation's types. In terms of the expected standard deviation, OLS matched the jackknife approach, but the minimum expected value of the estimators across replications of the jackknife was closer to the true value than the OLS regressions.

In the final simulations with n=500 observations, Bootstrap approach performed better than the rest of the estimation's types in terms of the accuracy of the estimator, a relative bias of 0.002% regarding from the true parameter was calculated with this approach. Also, the minimum expected value of the estimators was closer from this approach than the others, suggesting that bootstrap might be more appropriate for large samples.

According to the last results and as it is suggested by Speed (1994), researchers should perform also jackknife and bootstrap approaches when they're analyzing relationships from a set of variables in the multivariate regression framework, this in order to obtain more accurate estimations. However, the statistical significance might not be a good idea to be checked with the bootstrap approach since from this study, it was proved that its sensitive to the size of samples and might induce to type 1 errors more easily. Jackknife approach seems to be the most reliable method to perform correct inferences when the sample size is small.

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